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串　田　裕　彦

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Hirohiko　Kushida

海上保安大学校

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【Paper】

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Hirohiko Kushida [1]

Abstract

A direct cut-elimination procedure is proposed, which works for the sequent calculi not only of the classical and the intuitionistic logics but also of the standard modal logics. In the procedure, we do not introduce the inference rule *mix* but eliminate the usual rule *cut* directly. We demonstrate the method to establish the cut-elimination theorem for the hyper sequent calculus for the modal logic S5.

Keywords: Proof Theory, Sequent Calculus, Cut-elimination, Classical logic, Modal Logic

1 Introduction

The cut-elimination theorem is the fundamental theorem on proof theory in mathematical logic. It is a sort of normalization theorem on proofs in logical systems. The normal form of a proof, that is, a proof containing no inference rule *cut* in the style of sequent calculus, has an important property called the subformula property: every formula occurring in a proof is a subformula of the end-formula of the proof. This property is useful to derive some basic properties of a proof system including consistency of the proof system.

The cut-elimination theorem was originally established by Gentzen [8, 9] in the 1930's for the sequent calculi for the classical predicate logic and the intuitionistic predicate one called LK and LJ, respectively. Further Gentzen established consistency for a formal system of elementary number theory based on consistency of LK by extending a finitary method of proof theory in a specific way.

Gentzen introduced the rule called *mix* and considered the sequent calculi where *mix* is substituted for *cut* to facilitate the proof of the cut-elimination for LK and LJ. The rule *mix* is a generalization of *cut*, and it is enough to provide a procedure of mix-elimination for proofs to establish the cut-elimination. Then it is quite natural that there have been a line of studies on the direct cut-elimination: how to eliminate *cut* directly without introducing *mix* (Szabo [20], Buss [5], Borisavljević [2–4], von Plato [19]).

The methods of Borisavljević [2–4] are basically certain refinements of Gentzen's original proof-transformation; they are based on the method of lifting up a fixed application of *cut* by permuting it with the application of the inference rule just above the *cut* one by one, until the above application is the initial sequent or the rule *weakening* introducing the cut formula. We call this kind of method the *permutation* method. Gentzen developed this method by using *mix* instead of *cut*.

There is another method of the direct cut-elimination, which we call the *formula-reduction* method. In Buss [5] the cut-elimination is executed by reducing the cut formula to an atomic formula or the smallest sub-formula introduced in the proof under consideration. Here, the position of a fixed application of *cut* is not moved in a given proof. In von Plato [19], the permutation method appears to be basically taken; however, the crucial part of the transformation is concerned with the application of the rule *contraction* to be permuted with the *cut* and then the method of formula-reduction is taken for this part.

In the present paper, another method of the direct cut-elimination is proposed, which we shall call *topdown.* By this method we lift up a fixed application of *cut* all at once so that the cut formula is a sub-formula of the original cut formula. Then, it is verified that the inference rules applied between the new and old locations of *cut* are preserved, though this verification is straightforward.

An advantage of this topdown method is, as we

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[1]Japan Coast Guard Academy,　hkushida@jcga.ac.jp

demonstrate below, that it can be applied to a wide range of non-classical logic, in particular, modal logic. It is not clear if the former two methods work for modal logic systems, where there are characteristic rules for modality which depend on environment of sequents. For example, it is easy to see that the formula-reduction method does not work for most of modal logic systems. Suppose that we are given a proof ending with the following *cut*.

$$\frac{\Gamma \Rightarrow \Delta, B \wedge C \qquad B \wedge C, \Pi \Rightarrow \Theta}{\Gamma, \Pi \Rightarrow \Delta, \Theta}$$

In the formula-reduction method, the applications of inference rules introducing the cut formula are deleted with some modifications on structural rules to obtain the proofs of

$$\Gamma \Rightarrow \Delta, B$$
$$\Gamma \Rightarrow \Delta, C$$
$$\text{and } B, C, \Pi \Rightarrow \Theta.$$

However, when the cut formula is a modal formula $\Box B$, in most systems of modal logic, it is not generally possible to obtain the proofs of

$$\Gamma \Rightarrow \Delta, B \text{ and } B, \Pi \Rightarrow \Theta.$$

This is due to the nature of inference rules for modality; there are certain constraints on auxiliary formulas in those rules. In particular, we can consider the case when the cut formula $\Box B$ in the right upper sequent of the *cut* is introduced by an application, say α, of rule on modality. Then it is possible that a descendant occurrence of $\Box B$ is an auxiliary formula of some applications of rules on modality between α and the *cut*. In such a case, when $\Box B$ is changed to B, those applications of rules on modality are not preserved between α and the *cut*.

On the other hand, the topdown method works for such cases as this. We demonstrate our method for a strong system of modal logic, S5. By this we intend to suggest that the method works for a wide range of modal logic systems like K, T, D, S4. Also, the logic S5 has been of a particular interest in the research community. In particular, the proof theoretic or syntactic analyses of S5 have attracted attention from researchers, apart from the recent treatments of the modal logics via hypersequent or other extended sequent. E.g. [6, 7, 11, 12, 14, 15, 17, 18, 21].

This paper is organized as follows. In §2, we introduce the modal logic called S5 with its axiomatic system, and formulate a hypersequent calculus for S5. In §3, the cut-elimination theorem for S5 is proven by the topdown method.

1.1 On a Paper by Mints

Kurokawa [13] provided the author an opportunity to take notice of a paper by Mints, after he wrote a final draft of the current paper. The paper [16] by Mints does not handle the subject: the direct cut-elimination, and, thus, there is no reference in [16] to the above-mentioned papers on the subject: Szabo [20], Borisavljević [2,4], Borisavljević, Dosen and Petric [3], von Plato [19]. Anyway we must note that the method proposed in [16] is essentially the same as the "topdown method" in the current paper. However, we claim that the following points are still certain contributions of the current paper for the research subject.

- In Mints' proof, systems are adopted where the structural rules are built in the sequent. Therefore, Mints [16] essentially gave a cut-elimination procedure for the system with *mix*-rule. Thus, it is not yet clear whether the method is applicable for the direct cut-elimination. In the current paper we actually perform the method for the system explicitly containing structural rules to solve the problem of the direct cut-elimination.

- The subject of Mints [16] was to establish the cut-elimination for the sequent calculus of a special kind of modal logic, which is very interesting, the logic of continuous transformations of a topological space introduced by Artemov, Davoren and Nerode [1]. On the other hand, our target is a standard and basic modal logic, S5, and the result exposed in the current paper shows the applicability of the topdown method to the hypersequent calculus, an important and standard extention of sequent calculus. The hypersequent calculus for S5 is considered to be the most natural proof system for S5.

In addition, Mints' paper [16] explained another cut-elimination method for the classical propositional logic. This is essentially the same as the one we call the "formula-reduction" method due to Buss [5], which is not cited in [16].

2 Modal Logic S5

Modal logic is originally a discipline regarding logical inference involving modality of necessity and possibility. Nowadays modal logic has been actively studied as epistemic logic, temporal logic, and deontic logic, among others, with various interpretations of modality.

In this section, we review the axiomatic system for the modal logic S5 and then provide a definition of our hypersequent system for S5.

We use logical symbols $\neg$, $\wedge$, and $\vee$ for negation ('not'), conjunction ('and'), and disjunction ('or'), respectively. Also, the symbol $\Box$ is used as the modality of necessity. The formulas of modal logic are constructed in the following grammar.

$$A \to \bot|p|\neg A|A \wedge A|A \vee A|\Box A$$

Here, p is an atomic formula; $\bot$ is a constant to signify falsehood. In the grammar we do not take into consideration the other modality of possibility $\Diamond$ nor the other propositional connectives like implication $\to$, because they are defined in terms of $\wedge, \neg$ and $\Box$. In the following we use $\to$ freely.

The axiomatic system of the modal logic S5 is the classical propositional logic (of this language) extended with the inference rule called the necessitation rule: "$\vdash A$ implies $\vdash \Box A$" and the following axioms called $K, T, 4, 5$. [1]

K	$\Box(A \to B) \to (\Box A \to \Box B)$
T	$\Box A \to A$
4	$\Box A \to \Box\Box A$
5	$\neg\Box A \to \Box\neg\Box A$

Actually the 4-axiom is admissible in S5. Let us note that the 4- and 5-axioms are interesting in terms of epistemic interpretation. Consider a modality with an index, $\Box_a A$. It typically reads as "A is known to agent a".

$4 : \Box_a A \to \Box_a\Box_a A$
$5 : \neg\Box_a A \to \Box_a\neg\Box_a A$

Then, the two axioms express an agent's ability for the positive and negative introspection, respectively: "if an agent knows something, she knows that she knows it" and "if she does not know something, she knows that she does not know it".

[1]These are actually axiom schemata (and we do not adopt the substitution rule), although we just call them axioms.

2.1 Hypersequent Calculus for S5

We define sequents with the formula image f of them. When Γ and Δ are multisets of formulas, the form $\Gamma \Rightarrow \Delta$ is a *sequent*. Its formula image is defined by: $f(\Gamma \Rightarrow \Delta) = \Box(\bigwedge\Gamma \supset \bigvee\Delta)$.

Sequents are denoted by $S, T, U, \ldots$, possibly with integer subscripts. When $S_1, \ldots S_n$ are sequents, the form $S_1|S_2|\cdots|S_n$ is a hypersequent and its formula image is defined as follows.

$$f(S_1|S_2|\cdots|S_n) = f(S_1) \vee \cdots \vee f(S_n)$$

Hypersequents are denoted by $\mathcal{H}, \mathcal{I}, \ldots$ possibly with integer subscripts.

Hypersequent calculi are natural generalizations of sequent calculi. Several variations of hypersequent calculi for S5 and the cut-elimination for them are reviewed and analyzed in Bednarska and Indrzejczak [10]. Here, the hypersequent calculus for S5 we define below is another variant similar to Avron's and Kurokawa's systems. (We refer to [10] for these and other former systems.)

- Initial Sequents: $A \Rightarrow A \qquad \bot \Rightarrow$
- Logical Inference Rules:

$$\frac{\mathcal{H}|\Gamma \Rightarrow \Delta, A}{\mathcal{H}|\Gamma \Rightarrow \Delta, A \vee B} \vee : r \qquad \frac{\mathcal{H}|\Gamma \Rightarrow \Delta, B}{\mathcal{H}|\Gamma \Rightarrow \Delta, A \vee B} \vee : r$$

$$\frac{\mathcal{H}|A, \Gamma \Rightarrow \Delta \quad \mathcal{H}|B, \Gamma \Rightarrow \Delta}{\mathcal{H}|A \vee B, \Gamma \Rightarrow \Delta} \vee : l$$

$$\text{H—}\frac{\Gamma \Rightarrow \Delta, A \quad \mathcal{H}|\Gamma \Rightarrow \Delta, B}{\mathcal{H}|\Gamma \Rightarrow \Delta, A \wedge B} \wedge : r$$

$$\frac{\mathcal{H}|A, \Gamma \Rightarrow \Delta}{\mathcal{H}|A \wedge B, \Gamma \Rightarrow \Delta} \wedge : l \qquad \frac{\mathcal{H}|B, \Gamma \Rightarrow \Delta}{\mathcal{H}|A \wedge B, \Gamma \Rightarrow \Delta} \wedge : l$$

$$\frac{\mathcal{H}|A, \Gamma \Rightarrow \Delta}{\mathcal{H}|\Gamma \Rightarrow \Delta, \neg A} \neg : r \qquad \frac{\mathcal{H}|\Gamma \Rightarrow \Delta, A}{\mathcal{H}|\neg A, \Gamma \Rightarrow \Delta} \neg : l$$

- Internal Structural Inference Rules:

$$\frac{\mathcal{H}|A, A, \Gamma \Rightarrow \Delta}{\mathcal{H}|A, \Gamma \Rightarrow \Delta} ic \qquad \frac{\mathcal{H}|\Gamma \Rightarrow \Delta, A, A}{\mathcal{H}|\Gamma \Rightarrow \Delta, A} ic$$

$$\frac{\mathcal{H}|\Gamma \Rightarrow \Delta}{\mathcal{H}|A, \Gamma \Rightarrow \Delta} iw \qquad \frac{\mathcal{H}|\Gamma \Rightarrow \Delta}{\mathcal{H}|\Gamma \Rightarrow \Delta, A} iw$$

$$\frac{\mathcal{H}|\Gamma \Rightarrow \Delta, A \quad \mathcal{H}|A, \Pi \Rightarrow \Theta}{\mathcal{H}|\Gamma, \Pi \Rightarrow \Delta, \Theta} cut$$

- External Structural Inference Rules

$$\dfrac{\mathcal{H}}{\mathcal{H}|S}\ ew \qquad \dfrac{\mathcal{H}|\Gamma \Rightarrow \Delta|\Theta \Rightarrow \Pi}{\mathcal{H}|\Gamma, \Theta \Rightarrow \Delta, \Pi}\ merge$$

- Inference Rules for Modality

$$\dfrac{\mathcal{H}|\Box\Gamma \Rightarrow A}{\mathcal{H}|\Box\Gamma \Rightarrow \Box A}\ \Box : r \qquad \dfrac{\mathcal{H}|A, \Gamma \Rightarrow \Delta}{\mathcal{H}|\Box A, \Gamma \Rightarrow \Delta}\ \Box : r$$

$$\dfrac{\mathcal{H}|\Box A, \Gamma \Rightarrow \Delta}{\mathcal{H}|\Box A \Rightarrow |\Gamma \Rightarrow \Delta}\ move$$

Here, A in *cut* is called the *cut formula*.

3 Topdown Cut-Elimination

In this section, we establish the cut-elimination for $\mathsf{S5}$ by executing the topdown method.

Theorem 3. 1. *Let $\mathcal{H}$ be a hypersequent provable in the hypersequent calculus for $\mathsf{S5}$. Then, it is possible to construct a cut-free proof of $\mathcal{H}$ in it.*

Let P be any proof in the hypersequent calculus for $\mathsf{S5}$ where the only application of *cut* occurs in the last step in the following form.

$$\dfrac{\mathcal{H}|\Gamma \Rightarrow \Delta, A \qquad \mathcal{H}|A, \Pi \Rightarrow \Theta}{\mathcal{H}|\Gamma, \Pi \Rightarrow \Theta, \Delta}\ cut$$

We are going to eliminate this application of *cut* without increasing other applications of *cut* or changing the end-hypersequent.

Without loss of generality, we restrict the form of the initial sequent $A \Rightarrow A$ to $p \Rightarrow p$ with an atomic formula p.

Suppose that there are n- (and m-, respectively) many applications of the logical inference rule, *iw*, *ew*, or the initial sequent introducing the cut formula A on the left (and right, respectively) upper sequent of the *cut*.

Let $(Q_1, \ldots, Q_n)$ and $(R_1, \ldots, R_m)$, respectively, be the lists of the paths starting from the lower hypersequent of a rule introducing A and ending with the left and right, respectively, lower hypersequent of the *cut*.

Proceed by induction on the size of the cut formula A, that is, the number of occurrences of logical symbols and the modality $\Box$ in A.

Base Case 1. A is an atomic formula p.

Suppose that P has the following form.

$$\dfrac{\begin{array}{c} p \Rightarrow p \\ Q_i \quad \vdots \\ \mathcal{H}|\Gamma \Rightarrow \Delta, p \end{array} \qquad \begin{array}{c} p \Rightarrow p \\ R_j \quad \vdots \\ \mathcal{H}|p, \Pi \Rightarrow \Theta \end{array}}{\mathcal{H}|\Gamma, \Pi \Rightarrow \Delta, \Theta}\ cut$$

For each R_j, we make a replacement as follows.

- When R_j starts with $p \Rightarrow p$, replace it with the subproof of $\mathcal{H}|\Gamma \Rightarrow \Delta, p$.

$$p \Rightarrow p \qquad \rhd \qquad \begin{array}{c} \vdots \\ \mathcal{H}|\Gamma \Rightarrow \Delta, p \end{array}$$

- When R_j starts with *iw*, execute the following replacement.

$$\begin{array}{c} \vdots \\ \dfrac{\mathcal{J}_j|\Sigma_j \Rightarrow \Psi_j}{\mathcal{J}_j|p, \Sigma_j \Rightarrow \Psi_j}\ iw \end{array}$$

$$\triangledown$$

$$\begin{array}{c} \vdots \\ \dfrac{\mathcal{J}_j|\Sigma_j \Rightarrow \Psi_j}{\mathcal{J}_j|\mathcal{H}|\Gamma, \Sigma_j \Rightarrow \Delta, \Psi_j}\ iw, ew \end{array}$$

- When R_j starts with *ew*, execute the following replacement.

$$\begin{array}{c} \vdots \\ \dfrac{\mathcal{J}_j}{\mathcal{J}_j|p, \Sigma_j \Rightarrow \Psi_j}\ ew \end{array}$$

$$\triangledown$$

$$\begin{array}{c} \vdots \\ \dfrac{\mathcal{J}_j}{\mathcal{J}_j|\mathcal{H}|\Gamma, \Sigma_j \Rightarrow \Delta, \Psi_j}\ iw, ew \end{array}$$

Now, simulate all R_j's under these replacements. Then we obtain the original end-hypersequent, by adding applications of *merge* in the last part.

Base Case 2. A is an atomic formula $\bot$.

Suppose that P has the following form.

$$\begin{array}{c} \bot \Rightarrow \bot \\ Q_i \quad \vdots \end{array} \qquad \begin{array}{c} \bot \Rightarrow \\ R_j \quad \vdots \end{array}$$

$$\dfrac{\mathcal{H}|\Gamma \Rightarrow \Delta, \bot \qquad \mathcal{H}|\bot, \Pi \Rightarrow \Theta}{\mathcal{H}|\Gamma, \Pi \Rightarrow \Delta, \Theta}\ cut$$

For each Q_i, we make a replacement as follows.

• When Q_i starts with $\bot \Rightarrow \bot$, make a replacement as follows.

$$\bot \Rightarrow \bot \qquad \rhd \qquad \bot \Rightarrow$$

• When Q_i starts with iw, we make the following replacement.

$$\dfrac{\vdots}{\dfrac{\mathcal{K}_i|\Phi_i \Rightarrow \Xi_i}{\mathcal{K}_i|\Phi_i \Rightarrow \Xi_i, \bot}\ iw}$$

$$\triangledown$$

$$\dfrac{\vdots}{\mathcal{K}_i|\Phi_i \Rightarrow \Xi_i}$$

• When R_j starts with ew, execute the following replacement.

$$\dfrac{\vdots}{\dfrac{\mathcal{K}_i}{\mathcal{K}_i|\Phi_i \Rightarrow \Xi_i, \bot}\ ew}$$

$$\triangledown$$

$$\dfrac{\vdots}{\dfrac{\mathcal{K}_i}{\mathcal{K}_i|\Phi_i \Rightarrow \Xi_i}\ ew}$$

Now, simulate all Q_i's under these replacements to obtain the hypersequent $\mathcal{H}|\Gamma \Rightarrow \Delta$. By applying iw several times, we obtain the original end-hypersequent $\mathcal{H}|\Gamma, \Pi \Rightarrow \Delta, \Theta$.

In the *Induction Step*, we proceed according to the cases on the form of A. We treat two cases; other cases are similarly proved.

Case 1. $A = B \wedge C$. Suppose that P has the following form.

$$\dfrac{\mathcal{K}_i|\Phi_i \Rightarrow \Xi_i, B \quad \mathcal{K}_i|\Phi_i \Rightarrow \Xi_i, C}{\mathcal{K}_i|\Phi_i \Rightarrow \Xi_i, B \wedge C}\ \wedge : r$$

$$Q_i \ \vdots$$

$$\dfrac{\mathcal{J}_j|B, \Sigma_j \Rightarrow \Psi_j}{\mathcal{J}_j|B \wedge C, \Sigma_j \Rightarrow \Psi_j}\ \wedge : l$$

$$R_j \ \vdots$$

$$\dfrac{\mathcal{H}|\Gamma \Rightarrow \Delta, B \wedge C \qquad \mathcal{H}|B \wedge C, \Pi \Rightarrow \Theta}{\mathcal{H}|\Gamma, \Pi \Rightarrow \Delta, \Theta}\ cut$$

Take Q_1. We distinguish cases as follows.

Case a. When Q_1 starts with iw or ew, make the following replacement.

$$\dfrac{\mathcal{K}_1|\Phi_1 \Rightarrow \Xi_1}{\mathcal{K}_1|\Phi_1 \Rightarrow \Xi_1, B \wedge C} \qquad \rhd \qquad \dfrac{\mathcal{K}_1|\Phi_1 \Rightarrow \Xi_1}{\mathcal{H}|\mathcal{K}_1|\Phi_1, \Pi \Rightarrow \Xi_1, \Theta}$$

$$\dfrac{\mathcal{K}_1}{\mathcal{K}_1|\Phi_1 \Rightarrow \Xi_1, B \wedge C} \qquad \rhd \qquad \dfrac{\mathcal{K}_1}{\mathcal{H}|\mathcal{K}_1|\Phi_1, \Pi \Rightarrow \Xi_1, \Theta}$$

Case b. When Q_1 starts with $\wedge : r$. For each R_j, we make a replacement as follows.

• When R_j starts with $\wedge : l$, replace it with *cut* with $\mathcal{K}_1|\Phi_1 \Rightarrow \Xi_1, B$ or $\mathcal{K}_1|\Phi_1 \Rightarrow \Xi_1, C$.

$$\dfrac{\mathcal{J}_j|B, \Sigma_j \Rightarrow \Psi_j}{\mathcal{J}_j|B \wedge C, \Sigma_j \Rightarrow \Psi_j}$$

$$\triangledown$$

$$\dfrac{\dfrac{\mathcal{K}_1|\Phi_1 \Rightarrow \Xi_1, B}{\mathcal{J}_j|\mathcal{K}_1|\Phi_1 \Rightarrow \Xi_1, B} \quad \dfrac{\mathcal{J}_j|B, \Sigma_j \Rightarrow \Psi_j}{\mathcal{J}_j|\mathcal{K}_1|B, \Sigma_j \Rightarrow \Psi_j}}{\mathcal{J}_j|\mathcal{K}_1|\Phi_1, \Sigma_j \Rightarrow \Xi_1, \Psi_j}$$

$$\dfrac{\mathcal{J}_j|C, \Sigma_j \Rightarrow \Psi_j}{\mathcal{J}_j|B \wedge C, \Sigma_j \Rightarrow \Psi_j}$$

$$\triangledown$$

$$\dfrac{\dfrac{\mathcal{K}_1|\Phi_1 \Rightarrow \Xi_1, C}{\mathcal{J}_j|\mathcal{K}_1|\Phi_1 \Rightarrow \Xi_1, C} \quad \dfrac{\mathcal{J}_j|C, \Sigma_j \Rightarrow \Psi_j}{\mathcal{J}_j|\mathcal{K}_1|C, \Sigma_j \Rightarrow \Psi_j}}{\mathcal{J}_j|\mathcal{K}_1|\Phi_1, \Sigma_j \Rightarrow \Xi_1, \Psi_j}$$

• When R_j starts with iw, execute the following replacement.

$$\dfrac{\mathcal{J}_j|\Sigma_j \Rightarrow \Psi_j}{\mathcal{J}_j|B \wedge C, \Sigma_j \Rightarrow \Psi_j} \qquad \rhd \qquad \dfrac{\mathcal{J}_j|\Sigma_j \Rightarrow \Psi_j}{\mathcal{J}_j|\mathcal{K}_1|\Phi_1, \Sigma_j \Rightarrow \Xi_1, \Psi_j}$$

• When R_j starts with ew, execute the following replacement.

$$\dfrac{\mathcal{J}_j}{\mathcal{J}_j|B \wedge C, \Sigma_j \Rightarrow \Psi_j} \qquad \rhd \qquad \dfrac{\mathcal{J}_j}{\mathcal{J}_j|\mathcal{K}_1|\Phi_1, \Sigma_j \Rightarrow \Xi_1, \Psi_j}$$

In this way, for each $1 \leq j \leq m$, we obtain the hypersequent $\mathcal{J}_j|\mathcal{K}_1|\Phi_1, \Sigma_j \Rightarrow \Xi_1, \Psi_j$. Then, we simulate each R_j to obtain the hypersequent $\mathcal{H}|\mathcal{K}_1|\Phi_1, \Pi \Rightarrow \Xi_1, \Theta$.

$$\begin{array}{c}\mathcal{J}_j|\mathcal{K}_1|\Phi_1,\Sigma_j \Rightarrow \Xi_1,\Psi_j\\ R_j\ \vdots\\ \mathcal{H}|\mathcal{K}_1|\Phi_1,\Pi \Rightarrow \Xi_1,\Theta\end{array}$$

We illustrate below a simple case when $m=2$; $R_{1,2}$ correspond to the above first and third cases.

$$\begin{array}{c}\dfrac{\mathcal{J}_1|B,\Sigma_1 \Rightarrow \Psi_1}{\mathcal{J}_1|B\wedge C,\Sigma_1 \Rightarrow \Psi_1} \qquad \dfrac{\mathcal{J}_2}{\mathcal{J}_2|B\wedge C,\Sigma_2 \Rightarrow \Psi_2}\\ \vdots\\ \mathcal{H}|B\wedge C,\Pi \Rightarrow \Theta\end{array}$$

$$\triangledown$$

$$\begin{array}{c}\dfrac{\dfrac{\mathcal{K}_1|\Phi_1 \Rightarrow \Xi_1,B}{\mathcal{J}_1|\mathcal{K}_1|\Phi_1 \Rightarrow \Xi_1,B} \quad \dfrac{\mathcal{J}_1|B,\Sigma_1 \Rightarrow \Psi_1}{\mathcal{J}_1|\mathcal{K}_1|B,\Sigma_1 \Rightarrow \Psi_1}}{\mathcal{J}_1|\mathcal{K}_1|\Phi_1,\Sigma_1 \Rightarrow \Xi_1,\Psi_1} \qquad \dfrac{\mathcal{J}_2}{\mathcal{J}_2|\mathcal{K}_1|\Phi_1,\Sigma_2 \Rightarrow \Xi_1,\Psi_2}\\ \vdots\\ \mathcal{H}|\mathcal{K}_1|\Phi_1,\Pi \Rightarrow \Xi_1,\Theta\end{array}$$

Now, we repeat this procedure for $Q_2,\ldots,Q_n$ to obtain the subproofs of $\mathcal{H}|\mathcal{K}_i|\Phi_i,\Pi \Rightarrow \Xi_i,\Theta$ $(1\le i\le n)$. Then, we can simulate $Q_1,Q_2,\ldots,Q_n$ to obtain the original end-hypersequent. The transformation is depicted as follows.

$$\begin{array}{c}\mathcal{K}_i|\Phi_i \Rightarrow \Xi_i,B\wedge C\\ Q_i\ \vdots\\ \mathcal{H}|\Gamma \Rightarrow \Delta,B\wedge C\end{array}$$

$$\triangledown$$

$$\begin{array}{c}\mathcal{H}|\mathcal{K}_i|\Phi_i,\Pi \Rightarrow \Xi_i,\Theta\\ Q_i\ \vdots\\ \dfrac{\mathcal{H}|\mathcal{H}|\Gamma,\Pi \Rightarrow \Delta,\Theta}{\mathcal{H}|\Gamma,\Pi \Rightarrow \Delta,\Theta}\end{array}$$

Here, we add applications of *merge* in the last part to obtain the original end-hypersequent.

Case 2. $A=\Box B$. Suppose that P has the following form.

$$\dfrac{\begin{array}{c}\dfrac{\mathcal{K}_i|\Box\Phi_i \Rightarrow B}{\mathcal{K}_i|\Box\Phi_i \Rightarrow \Box B}\ \Box:r\\ Q_i\ \vdots\\ \mathcal{H}|\Gamma \Rightarrow \Delta,\Box B\end{array} \qquad \begin{array}{c}\dfrac{\mathcal{J}_j|B,\Sigma_j \Rightarrow \Psi_j}{\mathcal{J}_j|\Box B,\Sigma_j \Rightarrow \Psi_j}\ \Box:l\\ R_j\ \vdots\\ \mathcal{H}|\Box B,\Pi \Rightarrow \Theta\end{array}}{\mathcal{H}|\Gamma,\Pi \Rightarrow \Delta,\Theta}\ cut$$

Take Q_1. We distinguish cases as follows.

Case a. When Q_1 starts with *iw* or *ew*, make the following replacement.

$$\dfrac{\mathcal{K}_1|\Phi_1 \Rightarrow \Xi_1}{\mathcal{K}_1|\Phi_1 \Rightarrow \Xi_1,\Box B} \quad \triangleright \quad \dfrac{\mathcal{K}_1|\Phi_1 \Rightarrow \Xi_1}{\mathcal{H}|\mathcal{K}_1|\Phi_1,\Pi \Rightarrow \Xi_1,\Theta}$$

$$\dfrac{\mathcal{K}_1}{\mathcal{K}_1|\Phi_1 \Rightarrow \Xi_1,\Box B} \quad \triangleright \quad \dfrac{\mathcal{K}_1}{\mathcal{H}|\mathcal{K}_1|\Phi_1,\Pi \Rightarrow \Xi_1,\Theta}$$

Case b. When Q_1 starts with $\Box:r$. For each R_j, we make a replacement as follows.

- When R_j starts with $\Box:l$, replace it with *cut* with $\mathcal{K}_1|\Box\Phi_1 \Rightarrow \Box B$.

$$\dfrac{\mathcal{J}_j|B,\Sigma_j \Rightarrow \Psi_j}{\mathcal{J}_j|\Box B,\Sigma_j \Rightarrow \Psi_j}$$

$$\triangledown$$

$$\dfrac{\dfrac{\mathcal{K}_1|\Box\Phi_1 \Rightarrow B}{\mathcal{J}_j|\mathcal{K}_1|\Box\Phi_1 \Rightarrow B} \quad \dfrac{\mathcal{J}_j|B,\Sigma_j \Rightarrow \Psi_j}{\mathcal{J}_j|\mathcal{K}_1|B,\Sigma_j \Rightarrow \Psi_j}}{\mathcal{J}_j|\mathcal{K}_1|\Box\Phi_1,\Sigma_j \Rightarrow \Psi_j}$$

- When R_j starts with *iw*, execute the following replacement.

$$\dfrac{\mathcal{J}_j|\Sigma_j \Rightarrow \Psi_j}{\mathcal{J}_j|\Box B,\Sigma_j \Rightarrow \Psi_j} \quad \triangleright \quad \dfrac{\mathcal{J}_j|\Sigma_j \Rightarrow \Psi_j}{\mathcal{J}_j|\mathcal{K}_1|\Box\Phi_1,\Sigma_j \Rightarrow \Psi_j}$$

- When R_j starts with *ew*, execute the following replacement.

$$\dfrac{\mathcal{J}_j}{\mathcal{J}_j|\Box B,\Sigma_j \Rightarrow \Psi_j} \quad \triangleright \quad \dfrac{\mathcal{J}_j}{\mathcal{J}_j|\mathcal{K}_1|\Box\Phi_1,\Sigma_j \Rightarrow \Psi_j}$$

In this way, in *Case b*, for each $1\le j\le m$, we obtain the hypersequent $\mathcal{J}_j|\mathcal{K}_1|\Box\Phi_1,\Sigma_j \Rightarrow \Psi_j$. Then, we simulate each R_j to obtain the hypersequent $\mathcal{H}|\mathcal{K}_1|\Box\Phi_1,\Pi \Rightarrow \Theta$.

$$\begin{array}{c}\mathcal{J}_j|\mathcal{K}_1|\Box\Phi_1,\Sigma_j \Rightarrow \Psi_j\\ R_j\ \vdots\\ \mathcal{H}|\mathcal{K}_1|\Box\Phi_1,\Pi \Rightarrow \Theta\end{array}$$

Here, in this simulation, if there occur an application of *move*-rule moving $\Box B$ to another sequent, we must replace it with a series of *move*-rule each of which moves a modal formula from $\Box\Phi_1$.

We illustrate below a simple case when $m = 2$; $R_{1,2}$ correspond to the above first and second cases in *Case b*.

$$\dfrac{\mathcal{J}_1|B,\Sigma_1 \Rightarrow \Psi_1}{\mathcal{J}_1|\Box B,\Sigma_1 \Rightarrow \Psi_1} \qquad \dfrac{\mathcal{J}_2|\Sigma_2 \Rightarrow \Psi_2}{\mathcal{J}_2|\Box B,\Sigma_2 \Rightarrow \Psi_2}$$

$$\vdots$$

$$\mathcal{H}|\Box B,\Pi \Rightarrow \Theta$$

$$\bigtriangledown$$

$$\dfrac{\dfrac{\mathcal{K}_1|\Box\Phi_1 \Rightarrow B}{\mathcal{J}_1|\mathcal{K}_1|\Box\Phi_1 \Rightarrow B} \quad \dfrac{\mathcal{J}_1|B,\Sigma_1 \Rightarrow \Psi_1}{\mathcal{J}_1|\mathcal{K}_1|B,\Sigma_1 \Rightarrow \Psi_1}}{\mathcal{J}_1|\mathcal{K}_1|\Box\Phi_1,\Sigma_1 \Rightarrow \Psi_1} \qquad \dfrac{\mathcal{J}_2}{\mathcal{J}_2|\mathcal{K}_1|\Box\Phi_1,\Sigma_2 \Rightarrow \Psi_2}$$

$$\vdots$$

$$\mathcal{H}|\mathcal{K}_1|\Box\Phi_1,\Pi \Rightarrow \Theta$$

Now, we repeat this procedure for $Q_2,\dots,Q_n$ to obtain the subproofs of $\mathcal{H}|\mathcal{K}_i|\Phi_i,\Pi \Rightarrow \Xi_i,\Theta$ or $\mathcal{H}|\mathcal{K}_i|\Box\Phi_i,\Pi \Rightarrow \Theta$ $(1 \le i \le n)$. Then, we can simulate $Q_1,Q_2,\dots,Q_n$ to obtain the original end-hypersequent. We illustrate a simple case $n = 2$; Q_1 and Q_2 correspond to *Case a* and *Case b*, respectively.

$$\begin{array}{ll} \mathcal{K}_1|\Phi_1 \Rightarrow \Xi_1,\Box B & \mathcal{K}_2|\Box\Phi_1 \Rightarrow \Box B \\ Q_1 & Q_2 \end{array}$$

$$\vdots$$

$$\mathcal{H}|\Gamma \Rightarrow \Delta,\Box B$$

$$\bigtriangledown$$

$$\begin{array}{ll} \mathcal{H}|\mathcal{K}_1|\Phi_1,\Pi \Rightarrow \Xi_1,\Theta & \mathcal{H}|\mathcal{K}_2|\Box\Phi_1,\Pi \Rightarrow \Theta \\ Q_1 & Q_2 \end{array}$$

$$\vdots$$

$$\dfrac{\mathcal{H}|\mathcal{H}|\Gamma,\Pi \Rightarrow \Delta,\Theta}{\mathcal{H}|\Gamma,\Pi \Rightarrow \Delta,\Theta}$$

Here, we add applications of *merge* in the last part to obtain the original end-hypersequent. ■

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