On Lie Algebras of the Derivations of Rational Function Algebras

【Paper】

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Naoki Kawamoto*

Abstract

The Lie algebras of the derivations of commutative algebras of rational functions are considered. It is well known that the derivation algebras of the polynomial algebras and the Laurent polynomial algebras are simple Lie algebras. It is shown that the similar results hold for the cases of algebras of rational functions containing the Laurent polynomials, and that there are representations of a class of abstract Lie algebras in the derivations of the algebras.

Keywords: Lie algebra, simple, derivation, rational function, representation

1 Introduction

In this paper the algebras are considered over a field F of characteristic zero. Let A be a commutative associative algebra. Then it is well known that the set of derivations $\mathrm{Der}(A)$ of A is a Lie algebra by the bracket product

$$[a\partial_1, b\partial_2] = a\partial_1(b)\partial_2 - b\partial_2(a)\partial_1 \qquad (a, b \in A,\ \partial_1, \partial_2 \in \mathrm{Der}(A)).$$

The structure of $\mathrm{Der}(A)$ is investigated by many authors. It is well known that $\mathrm{Der}(A)$ is simple for $A = F[x],\ F[x^{\pm 1}]$ (e.g. [3]). The general conditions for $\mathrm{Der}(A)$ to be simple are considered in [2] and [5].

V. I. Arnold [1] suggests considering the algebra $\mathbb{C}[x^{\pm 1}, (1-x)^{-1}]$ after $\mathbb{C}[x]$ and $\mathbb{C}[x^{\pm 1}]$. We follow his suggestion and consider the Lie algebra of derivations $L = \mathrm{Der}(F[x^{\pm 1}, (1-x)^{-1}])$, and see that L is simple, which can be shown also by the results of [2] and [5]. We show that L is obtained as a natural homomorphic image of an abstract Lie algebra. These results can be extended to multivariable cases.

2 Simplicity

Let $F[x^{\pm 1}, (1-x)^{-1}]$ be an algebra of rational functions in an indeterminate x. Then it is easy to see that the set of derivations of $F[x^{\pm 1}, (1-x)^{-1}]$

$$D = \mathrm{Der}(F[x^{\pm 1}, (1-x)^{-1}])$$

is a Lie algebra with basis elements

$$w_{n,m} = \frac{x^n}{(1-x)^m}\partial \qquad (n, m \in \mathbb{Z},\ \partial = \frac{d}{dx}),$$

and the bracket product

$$\begin{aligned}
&[w_{n,m}, w_{k,\ell}] \\
&= \frac{x^n}{(1-x)^m}\partial\left(\frac{x^k}{(1-x)^\ell}\right)\partial - \frac{x^k}{(1-x)^\ell}\partial\left(\frac{x^n}{(1-x)^m}\right)\partial \\
&= (k-n)\frac{x^{n+k-1}}{(1-x)^{m+\ell}}\partial + (\ell-m)\frac{x^{n+k}}{(1-x)^{m+\ell+1}}\partial \\
&= (k-n)w_{n+k-1,m+\ell} + (\ell-m)w_{n+k,m+\ell+1} \qquad (n, m, k, \ell \in \mathbb{Z}).
\end{aligned}$$

We note that the subspace of D spanned by $\{w_{n,0} \mid n \in \mathbb{Z}\}$ is denoted by $W_{\mathbb{Z}}$ in [3] and is known to be simple.

Proposition 1 The Lie algebra D is simple.

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*Japan Coast Guard Academy, kawamoto@jcga.ac.jp

Proof. Let K be a non-zero ideal of D, and α be a non-zero element of K. Let m be a sufficiently small negative integer. Then we can delete the terms containing $(1-x)^{-1}$ in the coefficient of $[\alpha, w_{n,m}]$. Hence

$$[\alpha, w_{n,m}] \in W_{\mathbb{Z}}.$$

Since $W_{\mathbb{Z}}$ is simple and K is an ideal of D, we have $W_{\mathbb{Z}} \subseteq K$, and $K = D$. That is, D is simple.

We remark that Proposition 1 and the following Proposition 2 can be obtained by using the results of [2] and [5].

2 An isomorphism

Let L be a vector space with a basis $\{e_{n,m} \mid n, m \in \mathbb{Z}\}$. We define the product of basis elements by

$$\begin{aligned}&[e_{n,m}, e_{k,\ell}] \\ &\quad = (k-n)e_{n+k-1,m+\ell} + (\ell - m)e_{n+k,m+\ell+1}.\end{aligned}$$

Then L is a Lie algebra called type L in [4]. Let

$$v_{n,m} = e_{n,m} - e_{n+1,m} - e_{n,m-1} \quad (n, m \in \mathbb{Z}),$$

and I be a subspace of L spanned by $\{v_{n,m} \mid n, m \in \mathbb{Z}\}$. Then

$$\begin{aligned}&[v_{n,m}, e_{k,\ell}] \\ &\quad = (k-n)v_{n+k-1,m+\ell} + (\ell - m)v_{n+k,m+\ell+1},\end{aligned}$$

and it follows that I is an ideal of L.

Now we consider a linear mapping φ form L to D defined by

$$\varphi(e_{n,m}) = w_{n,m} \quad (n, m \in \mathbb{Z}).$$

It is clear that φ is a representation of L. Since

$$\begin{aligned}&\varphi(v_{n,m}) \\ &\quad = w_{n,m} - w_{n+1,m} - w_{n,m-1} \\ &\quad = \frac{x^n}{(1-x)^m}\partial - \frac{x^{n+1}}{(1-x)^m}\partial - \frac{x^n}{(1-x)^{m-1}}\partial \\ &\quad = 0,\end{aligned}$$

we have

$$I \subseteq \mathrm{Ker}\varphi.$$

Theorem 1 The ideal I is equal to $\mathrm{Ker}\varphi$, and L/I is isomorphic to D. In particular, I is a maximal ideal of L.

Proof. For any $v \in L$ we write $\overline{v} = v + I \in L/I$. Then we have

$$\overline{v_{n,m}} = \overline{e_{n,m}} - \overline{e_{n+1,m}} - \overline{e_{n,m-1}} = \overline{0}.$$

Hence

$$\overline{e_{n,m-1}} = \overline{e_{n,m}} - \overline{e_{n+1,m}}.$$

By using this equation for $m < 0$ the element $\overline{e_{n,m}}$ is a linear combination of $\{\overline{e_{n,0}} \mid n \in \mathbb{Z}\}$. From

$$\overline{e_{n+1,m}} = \overline{e_{n,m}} - \overline{e_{n,m-1}}$$

the element $\overline{e_{n,m}}$ $(n, m > 0)$ is a linear combination of $\{\overline{e_{0,m}},\ \overline{e_{n,0}} \mid m, n \in \mathbb{N}\}$. By using the equation

$$\overline{e_{n,m}} = \overline{e_{n+1,m}} + \overline{e_{n,m-1}}$$

the element $\overline{e_{n,m}}$ $(n < 0,\ m > 0)$ can be shown to be a linear combination of $\{\overline{e_{0,m}},\ \overline{e_{n,0}} \mid m \in \mathbb{N},\ n \in \mathbb{Z}\}$.

Let S be a subspace of L spanned by the elements $\{e_{n,0},\ e_{0,m} \mid n \in \mathbb{Z},\ m \in \mathbb{N}\}$. Then from the above argument we have

$$L = S + I.$$

Since

$$\varphi(e_{n,0}) = x^n\partial, \quad \varphi(e_{0,m}) = \frac{1}{(1-x)^m}\partial,$$

we have $S \cap \mathrm{Ker}\varphi = 0$. Then it is easy to see (by the modular law) that $I = \mathrm{Ker}\varphi$, and L/I is isomorphic to D. Since D is simple, it follows that I is a maximal ideal of L.

3 Two indeterminates

We consider the case of two indeterminates x, y, that is, the algebra $F[x^{\pm 1}, (1-x)^{-1}, y^{\pm 1}, (1-y)^{-1}]$. The derivation algebra

$$D_2 = \mathrm{Der}(F[x^{\pm 1}, (1-x)^{-1}, y^{\pm 1}, (1-y)^{-1}])$$

is generated by

$$w_{M,i} = a(M)\partial_i \quad (M \in M_2(\mathbb{Z}),\ i \in \{1, 2\}),$$

where $M_2(\mathbb{Z})$ is the set of 2×2 matrices of the integers, and

$$a(M) = \frac{x^{m_{11}}}{(1-x)^{m_{12}}}\frac{y^{m_{21}}}{(1-y)^{m_{22}}},$$

$$\partial_1 = \frac{\partial}{\partial x},\ \partial_2 = \frac{\partial}{\partial y},$$

for

$$M = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}.$$

Let $E_{ij} \in M_2(\mathbb{Z})$ be the matrix having 1 in the (i, j) position and 0 elsewhere. Then we have

$$\partial_i(a(M)) = m_{i1}a(M - E_{i1}) + m_{i2}a(M + E_{i2}),$$

and the bracket product is as follows:

$$\begin{aligned}
&[w_{M,i},\ w_{N,j}] \\
&\quad = a(M)\partial_i\,(a(N))\,\partial_j - a(N)\partial_j\,(a(M))\,\partial_i \\
&\quad = a(M)(n_{i1}a(N - E_{i1}) \\
&\qquad + n_{i2}a(N + E_{i2}))\partial_j \\
&\qquad - a(N)(m_{j1}a(M - E_{j1}) \\
&\qquad - m_{j2}a(M + E_{j2}))\partial_i \\
&\quad = n_{i1}a(M + N - E_{i1})\partial_j \\
&\qquad - m_{j1}a(M + N - E_{j1})\partial_i \\
&\qquad + n_{i2}a(M + N + E_{i2}))\partial_j \\
&\qquad - m_{j2}a(M + N + E_{j2}))\partial_i \\
&\quad = n_{i1}w_{M+N-E_{i1},j} - m_{j1}w_{M+N-E_{j1},i} \\
&\qquad + n_{i2}w_{M+N+E_{i2},j} - m_{j2}w_{M+N+E_{j2},i}
\end{aligned}$$

$$(M = (m_{pq}), N = (n_{pq}) \in M_2(\mathbb{Z}),\ i, j \in \{1, 2\}).$$

Proposition 2 D_2 is a simple Lie algebra.

Proof. Let K be a non-zero ideal of D_2 with an element $\alpha \neq 0$. If we take the matrix M with sufficiently small negative integers m_{12}, m_{22}, then we can delete the terms $(1-x)^{-1}$, $(1-y)^{-1}$ in the coefficients of $[\alpha,\ w_{M,i}]$. That is,

$$[\alpha,\ w_{M,i}] \in W_{\mathbb{Z}\times\mathbb{Z}}.$$

Since $W_{\mathbb{Z}\times\mathbb{Z}}$ is simple ([3]), it follows that $K = D_2$, and D_2 is simple.

4 An extension

We can construct an extension of D_2 as in the case of D. Let L_2 be a vector space with basis $\{e_{M,i} \mid M \in M_2(\mathbb{Z}),\ i \in \{1, 2\}\}$. We can define the bracket product as

$$\begin{aligned}
&[e_{M,i},\ e_{N,j}] \\
&\quad = n_{i1}e_{M+N-E_{i1},j} - m_{j1}e_{M+N-E_{j1},i} \\
&\qquad + n_{i2}e_{M+N+E_{i2},j} - m_{j2}e_{M+N+E_{j2},i}
\end{aligned}$$

$$(M = (m_{pq}), N = (n_{pq}) \in M_2(\mathbb{Z}),\ i, j \in \{1, 2\}).$$

Then it is easy to see that L_2 is a Lie algebra. Let

$$v_{M,i} = e_{M,i} - e_{M+E_{i1},i} - e_{M-E_{i2},i} \qquad (M \in M_2(\mathbb{Z}),\ i \in \{1, 2\}),$$

then we have

$$\begin{aligned}
&[v_{M,i},\ e_{N,j}] \\
&\quad = n_{i1}v_{M+N-E_{i1},j} - m_{j1}v_{M+N-E_{j1},i} \\
&\qquad + n_{i2}v_{M+N+E_{i2},j} - m_{j2}v_{M+N+E_{j2},i}
\end{aligned}$$

$$(M = (m_{pq}), N = (n_{pq}) \in M_2(\mathbb{Z}),\ i, j \in \{1, 2\}).$$

Let I_2 be a subspace of L_2 spanned by

$$\{v_{M,i} \mid M \in M_2(\mathbb{Z}),\ i \in \{1, 2\}\}.$$

Then it follows that I_2 is an ideal of L_2. Let S_2 be a subspace of L_2 spanned by

$$\{e_{nE_{i1},j},\ e_{mE_{i2},j} \mid n \in \mathbb{Z},\ m \in \mathbb{N},\ i, j \in \{1, 2\}\}.$$

By the argument similar to the proof of Theorem 1 it is easy to see that

$$L_2 = S_2 + I_2.$$

Then we have the following theorem.

Theorem 2 The Lie algebra L_2/I_2 is isomorphic to D_2.

5 Increasing sequences

If we consider a sequence of rational functions

$$\frac{1}{1-x},\ \frac{1}{2-x},\ \frac{1}{3-x},\ \ldots,$$

then we have a increasing sequence of algebras

$$\begin{aligned}
&\quad F[x^{\pm 1}, (1-x)^{-1}] \\
&< F[x^{\pm 1}, (1-x)^{-1}, (2-x)^{-1}] \\
&< F[x^{\pm 1}, (1-x)^{-1}, (2-x)^{-1}, (3-x)^{-1}] \\
&< \cdots.
\end{aligned}$$

If we consider derivation algebras of the above algebras, then we can extend Theorem 1.

We may take another direction by giving indeterminates

$$z,\ u,\ v,\ \ldots,$$

and the increasing sequence

$$\begin{aligned}
&\quad F[x^{\pm 1}, (1-x)^{-1}] \\
&< F[x^{\pm 1}, (1-x)^{-1}, y^{\pm 1}, (1-y)^{-1}] \\
&< F[x^{\pm 1}, (1-x)^{-1}, y^{\pm 1}, (1-y)^{-1}, z^{\pm 1}, (1-z)^{-1}] \\
&< \cdots.
\end{aligned}$$

Then we can generalize Theorem 2 to the cases of several indeterminates.

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